J. Fluid Mech. (1969), *vol.* 37, *part* 3, *pp.* 565–575
Printed in Great Britain

Particle collision efficiencies for a sphere

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(Received 5 July 1968)

Trajectories are calculated for small particles introduced upstream into a fluid flowing past a fixed sphere. Unseparated potential flow is taken as the velocity profile for the fluid, and the effect of gravity is included in the formulation when it acts along the axis of symmetry. Using a numerical procedure, particle trajectories which graze the sphere, and the corresponding collision efficiencies, are calculated for values of the Stokes number σ. When gravity is neglected, an analytic solution is obtained for large values of σ which is in good agreement with the numerical results for σ as low as 5. These results are compared with those of Sell (1931) and Langmuir & Blodgett (1946). When gravity is included, a critical value of the Stokes number σ_c is calculated for which no collisions occur until $\sigma > \sigma_c$.

1. Introduction

The problem of particle collisions in a gas stream is one of extreme importance in the mechanism of the removal of dust, smoke and mists in certain types of air-cleaning devices. A collision is caused primarily by the divergence of a particle from the gas streamlines due to its own inertia. It subsequently crosses the streamlines and collides with the obstacle which has caused the disturbance. The two main secondary causes are sedimentation, which plays an important part in the neighbourhood of the stagnation point, and interception, for relatively large particles, when the trajectory of the centre of the particle does not intersect the collecting surface. In the present analysis the object or collector is taken to be a sphere, and the latter of the secondary causes is eliminated by considering the particles to be very small.

The collection efficiency, E, is defined to be the ratio of the number of particles striking the collector to the number which would strike it if the streamlines were not diverted by the collector. If the particles are uniformly distributed in the gas and are of negligible radius compared with that of the sphere, then E is given by the ratio y_0^2, where y_0 is the radial distance from the axis upstream, made dimensionless with respect to the radius of the sphere, for the particle which just grazes the sphere.

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The way in which the gas flows around the collector will depend on the Reynolds number R based on the collector. When R is large, the streamlines do not begin to diverge until they are close to the collector, and the flow pattern corresponds to that of an inviscid fluid, except of course near to the surface. When R is small, the viscous forces have a more marked effect on the flow in spreading the disturbance caused by the collector further upstream. Thus the streamlines have a shallower gradient in this case, and reduce the chances of the particles colliding with the collector by diminishing the influence of their inertia. For Reynolds numbers greater than 1000 unseparated potential flow provides a fair approximation to the actual flow field near the forward surface of the collector, and for this reason it is used herein.

In this paper the authors have calculated the trajectories of small particles introduced upstream into a fluid flowing past a fixed sphere. When gravity is neglected, collision efficiencies are calculated for values of the Stokes number σ, and compared with the results of Sell (1931) and Langmuir & Blodgett (1946). When σ is large an analytic solution is derived which is exact to order σ^{-2}.

Gravity is included in the formulation when it acts along the axis of symmetry. When it acts in such a sense as to oppose the motion of the particles, σ_c, the critical value of the Stokes number for which no collision occurs until $\sigma > \sigma_c$, is calculated numerically. It is shown that no such σ_c exists when gravity acts in the other sense. Collision efficiencies are again calculated numerically for both cases.

2. Collision of particles with a sphere for values of $\sigma > \frac{1}{12}$

We consider a sphere of radius a moving steadily with speed U through a uniform unbounded fluid. If we assume an incompressible irrotational flow pattern about the sphere it was seen in Michael (1968) that the equation of motion of a small particle towards the sphere along the upstream axis of symmetry is, in dimensionless form,

$$\sigma v \frac{dv}{dr} = -\left\{v + \left(1 - \frac{1}{r^3}\right)\right\}, \tag{1}$$

where v is the velocity of the particle radially outwards, measured on the scale U, r is the distance from the centre of the sphere on the scale a, and σ is the Stokes number $\tau U/a$, where the relaxation time of the particle is τ. For spherical particles of radius d and density ρ_d, in a fluid of density ρ and kinematic viscosity ν,

$$\tau = \frac{2}{9}\frac{d^2\rho_d}{\nu\rho}, \quad \text{and} \quad \sigma = \frac{2R}{9}\left(\frac{\rho_d}{\rho}\right)\left(\frac{d}{a}\right)^2,$$

where R is the Reynolds number Ua/ν. The equation assumes that the particle is sufficiently small for a linear steady Stokes law of resistance between the particle and the fluid to be appropriate. It is also assumed that the resistance is opposed to the direction of motion of the particle relative to the fluid.

The influence of gravity on the trajectory of the particle may be conveniently measured by the ratio ϵ of the terminal velocity of free fall of the particle in the fluid to U. Thus, if g is the gravitational acceleration

$$\epsilon = \frac{2}{9}\frac{gd^2\rho_d}{\nu U\rho}.$$

For small enough particles this quantity can in general be neglected. For example with $\rho_d = 1$, $d = 10^{-3}$ cm in air the terminal velocity will be 1·28 cm/sec, so that provided only that $U \gg 1{\cdot}28$ cm/sec the gravitational effect can be neglected. We may also remark here on the justification for the neglect of the unsteady Stokes drag terms, and the inertial force. Compared with the steady Stokes force which is retained these effects are of the order $(Ud^2/a\nu)^{\frac{1}{2}}$ and $Ud^2/a\nu$ respectively, when the relative velocity of the particles to the gas flow is of order U. If we use as representative values $U = 10^3$ cm/sec and $\nu = 0{\cdot}15$ cm^2/sec for air these ratios are of the order $10^{-\frac{3}{2}}$ and 10^{-3}.

It has been shown by Langmuir & Blodgett (1946) (see also Michael 1968) that a particle moving according to (1) will reach the sphere $r = 1$ in a finite time, only when $\sigma > \frac{1}{12}$. When $\sigma > \frac{1}{12}$ a particle which is initially off the axis of symmetry and moving with the gas upstream may collide with the sphere, and it is of interest to establish the collision cross-section as a function of σ, the collision cross-section being the circle of radius y_0 about the axis such that a particle starting upstream from a position within this circle will collide with the sphere. Results are given here for collision cross sections based on numerical integration of particle trajectories using a trial and error method to establish the value of y_0, for a given $\sigma > \frac{1}{12}$, at which the trajectory just touches the sphere.

We write the fluid velocity $\mathbf{u} = \operatorname{grad}\phi$, where $\phi = (r + [1/2r^2])\cos\theta$. The velocity components of $\mathbf{u}$ are

$$u_x = 1 + \frac{y^2-2x^2}{2(x^2+y^2)^{\frac{5}{2}}}, \quad u_y = \frac{-3xy}{2(x^2+y^2)^{\frac{5}{2}}}, \tag{2}$$

in which $x = r\cos\theta$, and $y = r\sin\theta$.

We first examine the motion of the particle at a large distance upstream assuming that it moves with the gas velocity at $x = -\infty$. If v_x, v_y are the component velocities of the particle, equations of motion are

$$\sigma v_x \frac{dv_x}{dx} = u_x - v_x, \tag{3}$$

$$\sigma v_x \frac{dv_y}{dx} = u_y - v_y. \tag{4}$$

After substitution from (2) into (3) and (4) we can find a series expansion for v_x and v_y, in powers of $1/x$, suitable for use at large $|x|$. Thus

$$v_x = 1 + \frac{1}{x^3} + \frac{3\sigma}{x^4} + \frac{12}{x^5} + O\left(\frac{1}{x^6}\right),$$

$$v_y = \frac{3y}{2x^4} + \frac{6\sigma y}{x^5} + \frac{30\sigma^2 y - 15y^3/4}{x^6} + O\left(\frac{1}{x^7}\right).$$

We deduce from these equations that the particle trajectory is given initially by

$$y = y_0 - \frac{y_0}{2x^3} - \frac{3\sigma y_0}{2x^4} - \frac{(6\sigma^2 y_0 - 3y_0^3/4)}{x^5} + O\left(\frac{1}{x^6}\right).$$

This series does not converge well enough to be of use when $|x| \sim 1$. In our work it was used to give a starting-point at $x = -10$ for a numerical integration of (3) and (4). Step by step integration was carried out with the positions -10, (0·02), -2, (0·001), 0 in x. The smaller step length near $x = 0$ is necessary for the cases in which σ is just greater than $\frac{1}{12}$, because in such a case the critical trajectory passes near the stagnation point on the sphere where the velocities are small.

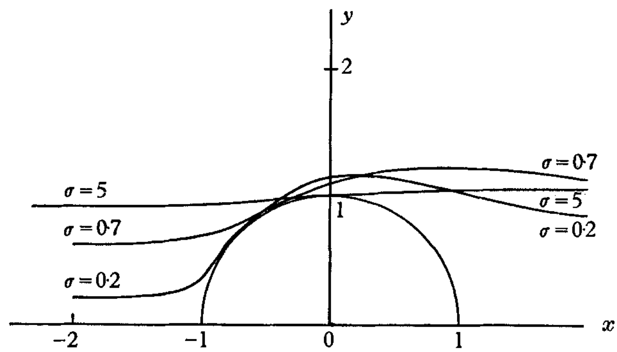

FIGURE 1. Grazing particle trajectories for values of σ = 0·2, 0·7, 5·0.

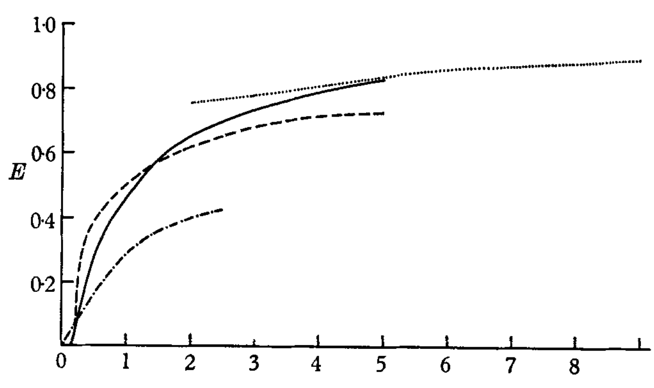

FIGURE 2. Collision efficiency E as a function of σ. ——, Michael & Norey; – – – –, Langmuir & Blodgett (1946); –·–·–·–, Sell (1931); · · · ·, analytical result for large σ.

Table 1 gives values of y_0 for grazing trajectories for different values of σ. Included in this table for comparison are values of y_0 obtained by Langmuir & Blodgett who also computed trajectories numerically starting at the position $x = -3$. Grazing trajectories obtained here are plotted in figure 1, and in figure 2 the collision efficiency ($= 1/\pi a^2$ (collision cross-sectional area)) is plotted against σ. This figure also includes results given by Sell (1931) and Langmuir & Blodgett (1946).

We conclude this section of the work with a discussion of particle trajectories when σ is large. If $s = 1/\sigma$, the equation of motion of a particle may be written

$$d\mathbf{v}/dt = s(\mathbf{u} - \mathbf{v}),$$

where $\mathbf{u}$, $\mathbf{v}$ are the gas and particle velocities respectively. When $s = 0$, $\mathbf{v}$ is constant for a particle and in our problem we would have $\mathbf{v} = U\hat{\mathbf{x}}$. For small s we write in dimensionless form

$$\mathbf{v} = \hat{\mathbf{x}} + s\mathbf{v}_1 + s^2\mathbf{v}_2 + \dots.$$

The first-order equation for $\mathbf{v}_1$ is then

$$\frac{d\mathbf{v}_1}{dx} = \operatorname{grad}\phi', \quad \text{where} \quad \phi' = \frac{\cos\theta}{2r^2}.$$

σ	y_0 Michael & Norey	y_0 Langmuir & Blodgett
0·1	0·011	—
0·2	0·186	$0{\cdot}2 < y_0 < 0{\cdot}25$
0·3	0·327	0·477
0·7	0·588	$0{\cdot}66 < y_0 < 0{\cdot}7$
2·0	0·807	$0{\cdot}78 < y_0 < 0{\cdot}8$
3·0	0·86	—
3·5	0·88	—
4·0	0·89	—
5·0	0·906	$0{\cdot}849 < y_0 < 0{\cdot}86$

TABLE 1. Values of y_0 for different σ.

This has a solution $\mathbf{v}_1 = -\frac{1}{2}\operatorname{grad}(1/r)$, which represents the velocity of a point source of strength $\frac{1}{2}$. The second-order equation for $\mathbf{v}_2$ is

$$\frac{d\mathbf{v}_2}{dx} = \operatorname{grad}\left(\frac{1}{2r} - \frac{1}{8r^4}\right),$$

which has a solution

$$\mathbf{v}_2 = \operatorname{grad}\left\{-\frac{1}{16r^3}\left(\frac{\cos\theta}{\sin^2\theta} - \frac{\theta}{\sin^3\theta}\right) + \tfrac{1}{2}\sinh^{-1}(\cot\theta)\right\} + \frac{1}{y}\left(\frac{3\pi}{16y^3} - \frac{1}{2}\right)\operatorname{grad} y.$$

When $s = 0$ the collision cross section has radius $y_0 = 1$, and we can find a power series in s showing how this is reduced when s is small. In spherical polar coordinates we have

$$v_r = \cos\theta + \frac{s}{2r^2} + s^2\left\{\frac{3}{16r^4}\left(\frac{\cos\theta}{\sin^2\theta} - \frac{\theta}{\sin^3\theta}\right) + \frac{\sin\theta}{y}\left(\frac{3\pi}{16y^3} - \frac{1}{2}\right)\right\} + O(s^3),$$

$$v_\theta = -\sin\theta + s^2\left\{-\frac{1}{2r\sin\theta} + \frac{1}{16r^4}\left(\frac{1}{\sin\theta} + \frac{1+2\cos^2\theta}{\sin^3\theta} - \frac{3\theta\cos\theta}{\sin^4\theta}\right) + \frac{\cos\theta}{y}\left(\frac{3\pi}{16y^3} - \frac{1}{2}\right)\right\} + O(s^3).$$

The particle path is given by

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{v_r}{v_\theta}. \tag{5}$$

When $s = 0$ the path is parallel to the axis and given by $r\sin\theta = y_0$. We write

$$r = \frac{y_0}{\sin\theta} + sl_1(\theta) + s^2 l_2(\theta),$$

to the second power in s. Substituting in (5) and equating like powers of s we find that
$$l_1(\theta) = \frac{1}{2y_0 \sin\theta}(\cos\theta + 1),$$
and
$$l_2(\theta) = -\frac{(1+\cos\theta)}{2\sin^2\theta} + \frac{1}{16y_0^3 \sin\theta}\{3(\pi-\theta)\cot\theta + \sin^2\theta - 4(\cos\theta+1)+3\}.$$
Thus the equation for the path becomes
$$r\sin\theta = y_0 + \frac{s}{2y_0}(\cos\theta+1) + s^2\left\{-\frac{(1+\cos\theta)}{2\sin\theta} + \frac{1}{16y_0^3} \times [3(\pi-\theta)\cot\theta + \sin^2\theta - 4(\cos\theta+1)+3]\right\}. \tag{6}$$
To arrive at the value of y_0 for which the path touches the sphere let $\theta = \theta_0$ at this position. Then $r = 1$ and $dr/d\theta = 0$ at $\theta = \theta_0$, and from (6) we have
$$y = 1 - \tfrac{1}{2}s + \tfrac{3}{8}s^2.$$
Thus the collision efficiency for large σ is given by
$$y_0^2 = 1 - \frac{1}{\sigma} + \frac{1}{\sigma^2} + O\left(\frac{1}{\sigma^3}\right).$$
This is shown in figure 2 and is in good agreement with the results obtained by numerical integration down to values of $\sigma \sim 5$.

3. The influence of gravity acting in the direction of motion of the sphere

In the preceding discussion particle trajectories were found on the assumption that the coefficient ϵ is small for very small particles. This means that gravity has only a small effect on a particle trajectory in which the particle velocity remains at all points of its path, of order U. However, particles which come close to the stagnation point of the sphere may be slowed down and for these paths gravity although represented in terms of a small coefficient can have a significant effect. This is particularly relevant when we are interested in the critical value, σ_c, of σ at which collisions will begin to occur. As we have seen previously these occur first at the stagnation point and, moreover, at the transition the particle velocity is zero and the time of approach is infinite. In this section of our paper we examine the modifications due to gravity in the case in which it acts in the direction of relative motion of the sphere through the gas, that is along the axis of symmetry. This case has the simplification that the axisymmetry is preserved.

Consider first the motion of a particle along the axis of symmetry upstream towards the sphere. The equation of motion may be written
$$\sigma v \frac{dv}{dr} = -\left\{v + (1 \pm \epsilon) - \frac{1}{r^3}\right\}. \tag{7}$$
When the sphere moves vertically upwards relative to the gas the positive sign is to be taken. The critical point of the equation is then moved to $r = (1+\epsilon)^{-\frac{1}{3}}$ which is within the sphere. In this case the particle will fall onto the sphere irrespective of the value of σ.

When the sphere moves vertically downwards the critical point moves to $r_0 = (1-\epsilon)^{-\frac{1}{3}}$ outside the sphere. If we write $r = (1-\epsilon)^{-\frac{1}{3}} + h$, (7) may be written

$$\sigma\, dv/dt = -[v + (3-4\epsilon)\, h], \quad dh/dt = v,$$

to the first power in ϵ, where t is the time. Hence

$$v = A\, e^{\lambda_1 t} + B\, e^{\lambda_2 t}, \quad h = \frac{A}{\lambda_1} e^{\lambda_1 t} + \frac{B}{\lambda_2} e^{\lambda_2 t},$$

in which A and B are constants, and λ_1, λ_2 are the roots of

$$\sigma\lambda^2 + \lambda + (3-4\epsilon) = 0.$$

Let
$$\sigma_0 = \frac{1}{4(3-4\epsilon)} = \tfrac{1}{12}(1 + \tfrac{4}{3}\epsilon + O(\epsilon^2)).$$

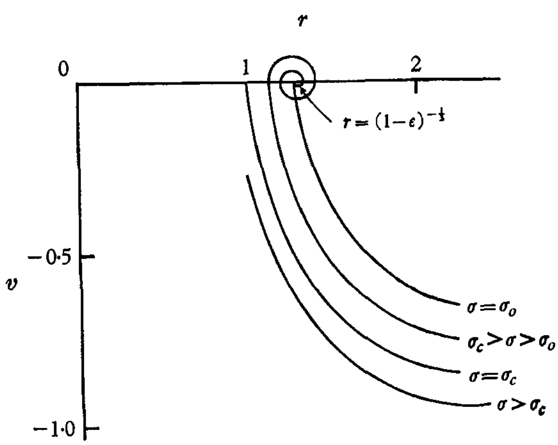

FIGURE 3. Sketch of particle paths on the upstream axis of symmetry.

When $\sigma < \sigma_0$, λ_1 and λ_2 are real roots and the particle approaches the critical point monotonically from upstream. When $\sigma > \sigma_0$ the particle spirals into the critical point and there is then the possibility of a collision with the sphere if on the first spiral it reaches the point $r = 1$. The particle paths in the (v, r) plane in these various cases are sketched in figure 3. We require to find the value of $\sigma_c > \sigma_0$ at which a spiral path just reaches $r = 1$. The critical condition will be that $v = 0$ when $r = 1$ or $h = -\frac{1}{3}\epsilon + O(\epsilon^2)$.

Hence
$$0 = A\, e^{\lambda_1 t} + B\, e^{\lambda_2 t}, \quad -\tfrac{1}{3}\epsilon = \frac{A}{\lambda_1} e^{\lambda_1 t} + \frac{B}{\lambda_2} e^{\lambda_2 t}.$$

If we let $t = 0$ correspond to this point

$$v = -\frac{\epsilon\lambda_1\lambda_2}{3(\lambda_2 - \lambda_1)} (e^{\lambda_1 t} - e^{\lambda_2 t}), \tag{8}$$

$$h = -\frac{\epsilon}{3(\lambda_2 - \lambda_1)} (\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}). \tag{9}$$

To obtain σ_c for a prescribed value of ϵ it is necessary to match the values of v and h given by (8) and (9) to an integral of (7) from upstream. As $r \to \infty$ $v \to -(1-\epsilon)$ in this case and a power series solution in terms of $1/r$ is given by

$$v = -(1-\epsilon) + \frac{1}{r^3} - \frac{3\sigma(1-\epsilon)}{r^4} + \frac{12(1-\epsilon)^2}{r^5} + \dots.$$

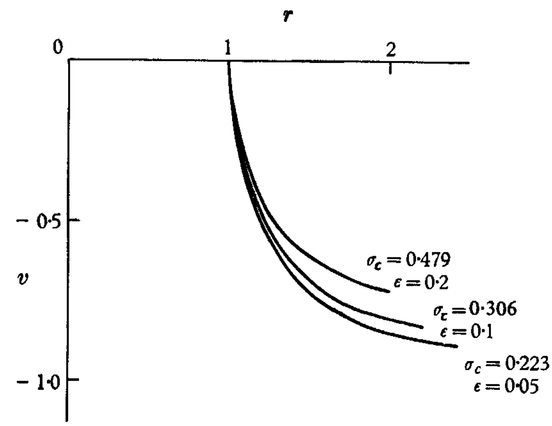

FIGURE 4. Critical particle paths on the upstream axis of symmetry are plotted for values of ϵ = 0·05, 0·1, 0·2.

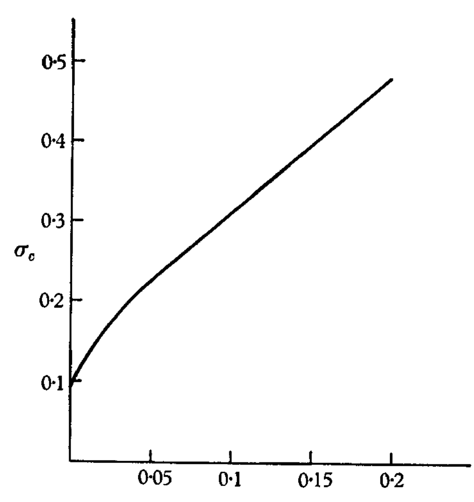

FIGURE 5. Critical value of σ is shown as a function of ϵ.

Our method was to start at $r = 10$ using the value of v given by this series for a trial value of σ. The equation was then integrated numerically using a step length of 10 (0·02) 2, and then 0·001 down to the point $r = 1+h$ with h given by (9) at $t = -2$. The value of v so obtained by integration can then be compared with that given by (8) at this value of t. By repeating this process for different values of σ we are able to identify the value of σ for which the values of v are the same. The calculation was done for $\epsilon = 0{\cdot}05$, 0·1 and 0·2, and the values of σ_c are 0·223, 0·306, 0·479 to three places of decimals respectively. Figure 4 shows the critical trajectories for these values of ϵ, and in figure 5, σ_c is shown as a function of ϵ.

The remaining results to report in this section concern the collision cross sections when $\epsilon \neq 0$, which have been obtained, again by numerical integration of the trajectories. The equations are in this case

$$\sigma v_x \frac{dv_x}{dx} = u_x - v_x \pm \epsilon, \tag{10}$$

$$\sigma v_x \frac{dv_y}{dx} = u_y - v_y. \tag{11}$$

The positive sign applies when the sphere moves upwards and the negative sign for downwards motion. We have seen that in the former case collisions occur when $\epsilon \neq 0$ for any σ and collision cross sections may be calculated for all values of σ in this case. The case $\sigma = 0$, $\epsilon \neq 0$ is easily dealt with mathematically because the

σ	y_0 for upward moving sphere			y_0 for downward moving sphere		
	$\epsilon = 0{\cdot}05$	$\epsilon = 0{\cdot}1$	$\epsilon = 0{\cdot}2$	$\epsilon = 0{\cdot}05$	$\epsilon = 0{\cdot}1$	$\epsilon = 0{\cdot}2$
0	0·05	0·09	0·17	No collision	No collision	No collision
0·3	0·40	0·46	0·54	0·22	No collision	No collision
0·7	0·63	0·66	0·71	0·54	0·49	0·31
2	0·82	0·84	0·86	0·79	0·76	0·70
5	—	—	—	—	0·89	0·86

TABLE 2. Values for y_0 for $\epsilon = 0{\cdot}05, 0{\cdot}10, 0{\cdot}2$.

particle velocity can be simply expressed. In fact the particle path can here be written in terms of the Stokes stream function as

$$\tfrac{1}{2}r^2(1+\epsilon-1/r^3)\sin^2\theta = k, \tag{12}$$

where k is constant. When we apply the conditions $r = 1$, $dr/d\theta = 0$ for a grazing trajectory we find $\theta = \frac{1}{2}\pi$, and $k = \frac{1}{2}\epsilon$, so that the collision cross section in this limit is given from (12) by $y_0^2 = 2k/(1+\epsilon) = \epsilon/(1+\epsilon)$. The numerical integration of (10) and (11) for larger values of σ yield the results given in table 2 and figure 6.

A similar numerical programme was performed for the downward moving sphere, and the collision efficiencies in this case are also shown in table 2 and figure 6.

4. Concluding remarks

It has been assumed in this work that particle trajectories calculated from equations applying at points away from the sphere boundary can be continued up to the spherical surface, and for this reason the results need qualification. Two effects which occur when a particle approaches the sphere are: (*a*) the effect of the viscous boundary layer, and (*b*) the change in the Stokes resistance due to the proximity of the wall. The first of these effects will arise when the particle comes within a distance $a/R^{\frac{1}{2}}$ of the boundary, where $R = Ua/\nu$. The second effect occurs when the distance is of order d. For small enough particles the region in which one has to consider the effects of (b) is a sublayer of the viscous boundary layer as for example with the data assumed in this paper, for which the viscous boundary layer is of order 10^{-2} cm. in thickness.

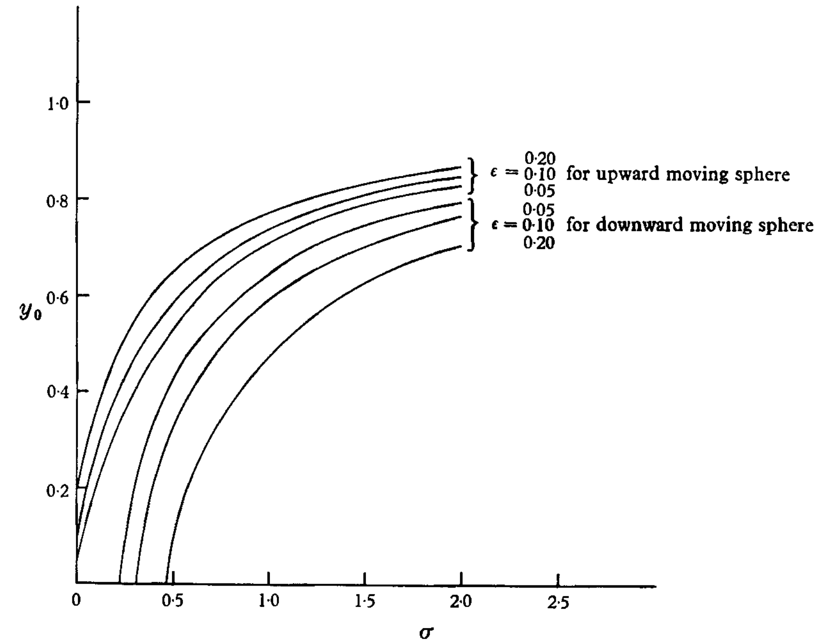

FIGURE 6. Value of y_0 as a function of σ, with $\epsilon = 0{\cdot}05,\ 0{\cdot}1,\ 0{\cdot}2$.

Provided that $R \gg 1$ the effect of a viscous boundary layer will be to change the gas velocity $\mathbf{u}$ in the boundary layer, and it is clear that, except in the cases where σ is small, discussed by Michael (1968), the change in the trajectories will be small and confined to the boundary layer. Thus although it requires further calculation to find the precise effects we do not expect the viscous boundary layer to produce any large changes in the results given in this paper.

When a particle comes within a distance of order d of the sphere the Stokes force arising from the particle motion normal to the sphere becomes proportional to the velocity/gap, the gap being the distance between the surface of the particle and the sphere (see Cox & Brenner 1967). Thus a calculation using this force will yield the result that, in the motion normal to the wall, a particle comes to rest before collision. Additional forces on the particle produced by the gas shear velocity in the boundary layer would mean that the particle continues to move very slowly towards the sphere but will not collide in a finite time. It seems clear that when dealing with very small particles a continuum model for the forces acting on a particle adjacent to the wall is inadequate, and that molecular surface forces become effective. Also it may be added that on the length scale of very small particle sizes the boundary of the sphere will no longer appear smooth. The conditions given in the paper should therefore be interpreted as conditions under which small particles may be brought to within a distance of the order of their radii of the surface of the sphere.

One further point of qualification concerns the cases of large σ for which collision cross sections become large. Grazing trajectories in these cases are not close to the stagnation point and it may be expected that changes in collision cross sections may occur in this case due to the effect of boundary layer separation, which produces increasing changes in the upstream velocity potential as one moves away from the stagnation point.

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